On the exponential sum with square–free numbers

D. I. Tolev

Introduction Consider the exponential sum

$$S(\alpha) = \sum_{n \le N} \mu^2(n)\, e(\alpha n),$$

where N is a large integer, $e(x) = e^{2\pi i x}$ and where $\mu(n)$ denotes the Möbius function. This sum appears naturally in the study of various problems involving squarefree numbers. For example, if $r_\nu(N)$ is the number of representations of N as the sum of ν square–free numbers, then we have

$$r_\nu(N) = \int_0^1 S(\alpha)^\nu\, e(-\alpha N)\, d\alpha\,.$$

In the case $\nu \ge 3$ Evelyn and Linfoot [5], Mirsky [7] and Brüdern and Perelli [2] used the circle method to obtain asymptotic formulae of the form

$$r_\nu(N) = \frac{1}{(\nu-1)!} \left(\frac{6}{\pi^2}\right)^\nu \mathfrak{S}_\nu(N)\, N^{\nu-1} + \Delta_\nu(N)\,, \qquad \Delta_\nu(N) = o(N^{\nu-1})\,,$$

where

$$\mathfrak{S}_\nu(N) = \prod_{p^2 \nmid N} \left(1 - \frac{1}{(1-p^2)^\nu}\right) \prod_{p^2 \mid N} \left(1 - \frac{1}{(1-p^2)^{\nu-1}}\right).$$

For $\nu \ge 3$ the sharpest estimate for $\Delta_\nu(N)$ is due to Brüdern and Perelli. They prove ([2], Theorem 1) that if $\nu \ge 3$ then $\Delta_\nu(N) = \mathcal{O}(N^{\nu-3/2+\varepsilon})$ holds, where here and later $\varepsilon > 0$ denotes an arbitrarily small number, which is not the same in different occurences.

Brüdern and Perelli also show that the last estimate is best possible up to the current knowledge about Riemann's zeta function $\zeta(s)$. More precisely

([2], Theorem 2), if $\nu \geq 2$ then $\Delta_\nu(N) = \Omega(N^{\nu-2+\theta/2-\varepsilon})$, where θ is the supremum of the real parts of the zeros of $\zeta(s)$. It is however established ([2], Theorem 3) that under the generalized Riemann hypothesis (GRH) for all Dirichlet L–functions one has $\Delta_\nu(N) = \mathcal{O}(N^{\nu-7/4+\varepsilon})$ for $\nu \geq 4$ and also $\Delta_3(N) = \mathcal{O}(N^{37/28+\varepsilon})$. Thus, in the case $\nu = 3$ they miss the optimal exponent by 1/14.

One of the key instruments for obtaining these results is the estimate of the sum $S(\alpha)$ on the set of minor arcs. Suppose that $Q \geq 1$ and denote

$$\mathcal{M}(Q) = \bigcup_{q \leq Q} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{Q}{qN}, \frac{a}{q} + \frac{Q}{qN}\right], \quad m(Q) = \left[\frac{Q}{N}, 1 + \frac{Q}{N}\right] \setminus \mathcal{M}(Q).$$

Theorem 4 of [2] states that if $Q \leq N^{3/7}$, then the estimate (2), given below, holds. A similar result was previously obtained by Baker, Brüdern and Harman [1], but under the condition $Q \leq N^{1/3}$.

The aim of the present paper is to present the proof of the following:

Theorem. *Suppose that*

$$1 \leq Q \leq N^{1/2} \tag{1}$$

and let $m(Q)$ be defined as above. Then for the sum $S(\alpha)$ we have

$$\sup_{\alpha \in m(Q)} |S(\alpha)| \ll N^{1+\varepsilon} Q^{-1}\,. \tag{2}$$

It is pointed out by Brüdern and Perelli at the end of section 5 of [2] that from this theorem one can obtain:

Corollary. *Suppose that GRH holds. Then we have $\Delta_3(N) = \mathcal{O}(N^{5/4+\varepsilon})$.*

We should mention that the circle method provides a non–trivial estimate for $\Delta_2(N)$ as well (see Brüdern *et al* [3]), but it is weaker then the estimate $\Delta_2(N) = \mathcal{O}(N^{2/3+\varepsilon})$, obtained by elementary methods by Evelyn and Linfoot [5] (see also Estermann [4] for simpler proof).

Furthermore, Heath–Brown [6] developed the *square sieve* and applied it to the related problem of counting square–free twins. Using his method one can obtain $\Delta_2(N) = \mathcal{O}(N^{7/11+\varepsilon})$.

To prove our theorem we apply the square sieve as well as some of the methods used in [2].

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Proof of the theorem Let us take arbitrary $\alpha \in m(Q)$. From Dirichlet's theorem and from the definition of $m(Q)$ it follows that there exist integers a and q satisfying

$$\left|\alpha - \frac{a}{q}\right| \le \frac{Q}{qN}, \qquad (a,q)=1, \qquad Q < q \le \frac{N}{Q}. \tag{3}$$

It is clear that

$$S(\alpha) = \sum_{d \le N^{1/2}} \mu(d) \sum_{m \le Nd^{-2}} e(\alpha d^2 m)$$

and therefore, according to Lemma 5.1 of [8], we have

$$S(\alpha) \ll \log N \max_{\frac{1}{2} \le D \le N^{1/2}} Y(\alpha, D), \tag{4}$$

where

$$Y(\alpha, D) = \sum_{D < d \le 2D} \min(ND^{-2}, ||\alpha d^2||^{-1}) \tag{5}$$

and where $||x||$ denotes the difference from x to the nearest integer.

Brüdern and Perelli [2] estimate $Y(\alpha, D)$ by three different methods and establish that

$$Y(\alpha, D) \ll N^{1+\varepsilon} Q^{-1} \tag{6}$$

unless $D^2 \in \left[NQ^{-1}, Q^{4/3}\right)$. If $Q \le N^{3/7}$ the later interval is empty, so (6) is true for all $D \in \left[\frac{1}{2}, N^{1/2}\right]$. This implies Theorem 4 of [2].

To prove out Theorem we shall estimate $Y(\alpha, D)$ using the square sieve. We can assume that

$$\frac{N}{Q} \le D^2 \le Q^{4/3} \tag{7}$$

because in the other cases the estimate (6) is established in [2].

Let P be a parameter, which we shall specify later. Now we assume only that

$$N^{\eta} \le P \le N^{1/2} \quad \text{for some constant} \quad \eta > 0. \tag{8}$$

Consider the function

$$\kappa(k) = \Big(\frac{\log P}{P} \sum_{\substack{P<p\le 2P \\ p\nmid q}} \Big(\frac{k}{p}\Big) \Big)^2 , \tag{9}$$

where the summation is taken over primes and $\big(\frac{\cdot}{p}\big)$ stands for the Legendre symbol. Obviously $\kappa(k) \ge 0$ for any integer k. Furthermore, we have

$$\kappa(k) \gg 1 \quad \text{if} \quad k = d^2 \quad \text{for some} \quad d \in (D, 2D] \, . \tag{10}$$

Indeed, in this case

$$\sum_{\substack{P<p\le 2P \\ p\nmid q}} \Big(\frac{k}{p}\Big) = \sum_{\substack{P<p\le 2P \\ p\nmid dq}} 1 \ge \sum_{P<p\le 2P} 1 - \nu(dq) \, ,$$

where $\nu(n)$ denotes the number of different prime factors of n. It is well–known that $\nu(n) \le 2\log n$, so (10) follows from Tchebyshev's prime number theorem and (8), (9). Using (5) and (10) we get

$$Y(\alpha, D) \ll \sum_{D^2<k\le 4D^2} \kappa(k) \, \min\big(M, \, ||\alpha k||^{-1}\big) \, , \tag{11}$$

where we have put

$$M = ND^{-2} \, . \tag{12}$$

We can now expand the function $\min(M, ||x||^{-1})$ into Fourier series, but it does not converge very fast and some extra efforts are needed to deal with the 'tail' of this series. To avoid this we take a smooth function $G(M, x)$, which is periodic with period one, satisfy

$$\min\big(M, \, ||x||^{-1}\big) \le G(M, x) \tag{13}$$

and which has Fourier expansion

$$G(M,x)=\sum_{n\in\mathbb{Z}} c_n\, e(nx) \tag{14}$$

with coefficients satisfying

$$c_n \ll \log M \tag{15}$$

and

$$\sum_{|n|>M^{1+\varepsilon}} |c_n| \ll M^{-A} \tag{16}$$

for any arbitrarily large constant $A>0$ (the constant in the $\ll$ – symbol depends on ε and A).

To construct such a function we take an infinitely differentiable function $\omega(x)$, which is supported and positive in the interval $(-1,1)$ and such that $\int_{-\infty}^{\infty}\omega(x)\,dx=1$. Then for any $\rho>0$ we define $\omega_\rho(x)=\rho^{-1}\omega(\rho^{-1}x)$ and consider the convolution

$$g(M,x,\rho)=\int_{-\infty}^{\infty}\min\big(M,\ ||t||^{-1}\big)\ \omega_\rho(x-t)\,dt\,.$$

We put $G(M,x)=2g\big(M,x,(10M)^{-1}\big)$. It is not difficult to see that the conditions (13) – (16) hold.

Having in mind (11) – (14) and (16) we find

$$Y(\alpha,D)\ll 1+|Z(\alpha,D)|\,, \tag{17}$$

where

$$Z(\alpha,D)=\sum_{D^2<k\le 4D^2}\kappa(k)\sum_{|n|\le M^{1+\varepsilon}} c_n\, e(\alpha kn)\,.$$

Now we apply (9) to get

$$Z(\alpha,D)=\Big(\frac{\log P}{P}\Big)^2\sum_{D^2<k\le 4D^2}\ \sum_{\substack{P<p,p'\le 2P\\ (pp',q)=1}}\Big(\frac{k}{pp'}\Big)\sum_{|n|\le M^{1+\varepsilon}} c_n\, e(\alpha kn)\,,$$

where $\big(\frac{\cdot}{pp'}\big)$ is the Jacobi symbol. We represent the last sum in the form

$$Z(\alpha,D)=\Big(\frac{\log P}{P}\Big)^2(\mathcal{E}_1+\mathcal{E}_2)\,, \tag{18}$$

where

$$\mathcal{E}_1 = \sum_{D^2<k\le 4D^2} \sum_{\substack{P<p\le 2P\\ p\nmid kq}} \sum_{|n|\le M^{1+\varepsilon}} c_n\, e(\alpha kn)\,,$$

$$\mathcal{E}_2 = \sum_{D^2<k\le 4D^2} \sum_{p,p'\,:\,(19)} \Big(\frac{k}{pp'}\Big) \sum_{|n|\le M^{1+\varepsilon}} c_n\, e(\alpha kn)$$

and where the summation over p and p' is taken over primes satisfying

$$P < p, p' \le 2P\,, \qquad (pp', q) = 1\,, \qquad p \ne p'\,. \tag{19}$$

Consider the sum $\mathcal{E}_1$. It is clear that

$$\mathcal{E}_1 = \sum_{0<|n|\le M^{1+\varepsilon}} c_n \sum_{\substack{P<p\le 2P\\ p\nmid q}} \sum_{\substack{D^2<k\le 4D^2\\ k\not\equiv 0 \pmod p}} e(\alpha kn) \,+\, \mathcal{O}(N^\varepsilon P D^2)\,. \tag{20}$$

According to Lemma 5.1 from [8], for the sum over k we have

$$\begin{aligned}\sum_{\substack{D^2<k\le 4D^2\\ k\not\equiv 0 \pmod p}} e(\alpha kn) &= \sum_{D^2<k\le 4D^2} e(\alpha kn) - \sum_{\substack{D^2<k\le 4D^2\\ k\equiv 0 \pmod p}} e(\alpha kn)\\ &\ll \min(D^2, ||\alpha n||^{-1}) + \min(D^2P^{-1}, ||\alpha p n||^{-1})\,.\end{aligned} \tag{21}$$

Hence from (12), (15), (20) and (21) we get

$$\mathcal{E}_1 \ll N^\varepsilon (P\mathcal{E}_1^{(1)} + \mathcal{E}_1^{(2)} + PD^2)\,, \tag{22}$$

where

$$\mathcal{E}_1^{(1)} = \sum_{n\le M^{1+\varepsilon}} \min(D^2, ||\alpha n||^{-1})\,,$$

$$\mathcal{E}_1^{(2)} = \sum_{n\le M^{1+\varepsilon}} \sum_{P<p\le 2P} \min(D^2P^{-1}, ||\alpha p n||^{-1})$$

(the summation is already taken over positive n only).

To estimate $\mathcal{E}_1^{(1)}$ we apply Lemma 5.4 of [8] and use (1), (3), (7) and (12) to get

$$\mathcal{E}_1^{(1)} \ll N^{1+\varepsilon}\Big(\frac{1}{q} + \frac{1}{D^2} + \frac{1}{M} + \frac{q}{N}\Big) \ll N^{1+\varepsilon}\Big(\frac{1}{Q} + \frac{D^2}{N}\Big)\,. \tag{23}$$

Furthermore, we have

$$\mathcal{E}_1^{(2)} \ll \sum_{h \le 2PM^{1+\varepsilon}} \tau(h)\, \min(D^2P^{-1}, ||\alpha h||^{-1})\,,$$

where $\tau(h)$ is the divisor function. Proceeding as above we find

$$\mathcal{E}_1^{(2)} \ll N^{\varepsilon} \sum_{h \le 2PM^{1+\varepsilon}} \min(D^2P^{-1}, ||\alpha h||^{-1}) \ll N^{1+\varepsilon} P\Big(\frac{1}{Q} + \frac{D^2}{N}\Big)\,. \tag{24}$$

From (22) – (24) we obtain

$$\mathcal{E}_1 \ll N^{1+\varepsilon} P\Big(\frac{1}{Q} + \frac{D^2}{N}\Big)\,. \tag{25}$$

Consider now $\mathcal{E}_2$. We have

$$\mathcal{E}_2 = \sum_{p,p'\,:\,(19)} \sum_{|n| \le M^{1+\varepsilon}} c_n\, F_n(\alpha)\,,$$

where

$$F_n(\alpha) = \sum_{D^2 < k \le 4D^2} \Big(\frac{k}{pp'}\Big)\, e(\alpha n k)\,. \tag{26}$$

We divide $\mathcal{E}_2$ into two parts:

$$\mathcal{E}_2 = \mathcal{E}_2^{(1)} + \mathcal{E}_2^{(2)}\,, \tag{27}$$

where $\mathcal{E}_2^{(1)}$ is the contribution of the terms with $n \ne 0$ and $\mathcal{E}_2^{(2)}$ comes from the terms with $n = 0$.

Using Pólya – Vinogradov's inequality (see, for example, Theorem 2.1 of [8]) we find

$$F_0(\alpha) = \sum_{D^2 < k \le 4D^2} \Big(\frac{k}{pp'}\Big) \ll \sqrt{pp'}\, \log N$$

and therefore, according to (12) and (15), we get

$$\mathcal{E}_2^{(2)} \ll N^{\varepsilon} P^3\,. \tag{28}$$

Consider $\mathcal{E}_2^{(1)}$. Bearing in mind (15) we find

$$\mathcal{E}_2^{(1)} \ll N^\varepsilon \sum_{p,p' : (19)} \sum_{n \le M^{1+\varepsilon}} |F_n(\alpha)| . \tag{29}$$

We shall estimate the sum $F_n(\alpha)$, defined by (26). According to (3), we write α in the form

$$\alpha = \frac{a}{q} + \beta , \qquad \text{where} \qquad (a,q) = 1 , \qquad Q < q \le \frac{N}{Q} , \qquad |\beta| \le \frac{Q}{qN} . \tag{30}$$

Write

$$G(t) = \sum_{D^2 < k \le t} \left(\frac{k}{pp'} \right) e\left(\frac{ank}{q} \right) .$$

Using Abel's formula and (12), (29) and (30) we find

$$\begin{aligned} F_n(\alpha) &= \sum_{D^2 < k \le 4D^2} \left(\frac{k}{pp'} \right) e\left(\frac{ank}{q} \right) e(\beta n k) \\ &= - \int_{D^2}^{4D^2} G(t) \, \frac{d}{dt} e(\beta n t) \, dt \, + \, G(4D^2) \, e(\beta N 4 D^2) \\ &\ll (1 + |\beta| n D^2) \max_{t \in [D^2, 4D^2]} |G(t)| \\ &\ll N^\varepsilon \max_{t \in [D^2, 4D^2]} |G(t)| . \end{aligned} \tag{31}$$

Consider the sum $G(t)$. We divide it into $\mathcal{O}\big(D^2 (pp'q)^{-1}\big)$ complete sums and at most one incomplete sum modulo $pp'q$. However, since $(pp', q) = 1$, for the complete sum we have

$$\begin{aligned} \sum_{k=1}^{pp'q} \left(\frac{k}{pp'} \right) e\left(\frac{ank}{q} \right) &= \sum_{u=1}^{pp'} \sum_{v=1}^{q} \left(\frac{uq + vpp'}{pp'} \right) e\left(\frac{an(uq + vpp')}{q} \right) \\ &= \left(\frac{q}{pp'} \right) \sum_{u=1}^{pp'} \left(\frac{u}{pp'} \right) \sum_{v=1}^{q} e\left(\frac{anpp'v}{q} \right) = 0 . \end{aligned}$$

Hence we can write $G(t)$ in the form

$$G(t) = \sum_{K_1 < k \le K_2} \left(\frac{k}{pp'} \right) e\left(\frac{ank}{q} \right) ,$$

where

$$0 \le K_2 - K_1 < pp'q\,. \tag{32}$$

We proceed in a standard manner to find

$$\begin{aligned} G(t) &= \sum_{k=1}^{pp'q} \sum_{K_1<h\le K_2} \frac{1}{pp'q} \sum_{-\frac{pp'q}{2}<s\le\frac{pp'q}{2}} e\Big(\frac{s(k-h)}{pp'q}\Big) \Big(\frac{k}{pp'}\Big)\, e\Big(\frac{ank}{q}\Big) \\ &= \sum_{-\frac{pp'q}{2}<s\le\frac{pp'q}{2}} \psi_s\, \theta_s\,, \end{aligned} \tag{33}$$

where

$$\psi_s = \frac{1}{pp'q} \sum_{K_1<h\le K_2} e\Big(\frac{-hs}{pp'q}\Big)\,, \qquad \theta_s = \sum_{k=1}^{pp'q} \Big(\frac{k}{pp'}\Big)\, e\Big(\frac{ank}{q} + \frac{sk}{pp'q}\Big)\,.$$

Using (32) we easily get

$$\psi_s \ll (1+|s|)^{-1}\,. \tag{34}$$

Consider θ_s. We have

$$\begin{aligned} \theta_s &= \sum_{u=1}^{pp'} \sum_{v=1}^{q} \Big(\frac{uq+vpp'}{pp'}\Big)\, e\Big(\frac{an(uq+vpp')}{q} + \frac{s(uq+vpp')}{pp'q}\Big) \\ &= \sum_{u=1}^{pp'} \Big(\frac{uq}{pp'}\Big)\, e\Big(\frac{su}{pp'}\Big) \sum_{v=1}^{q} e\Big(\frac{(anpp'+s)v}{q}\Big)\,. \end{aligned}$$

Hence

$$\theta_s = \begin{cases} q\big(\frac{qs}{pp'}\big)\, \gamma(pp') & \text{if} \quad anpp'+s \equiv 0 \pmod q\,, \\ 0 & \text{otherwise}\;, \end{cases} \tag{35}$$

where

$$\gamma(pp') = \sum_{u=1}^{pp'} \Big(\frac{u}{pp'}\Big)\, e\Big(\frac{u}{pp'}\Big)$$

is the Gauss sum.

Using (33) – (35) and the estimate $\gamma(pp') \ll \sqrt{pp'}$ (see, for example, Lemma 1.6 of [8]) we find

$$G(t) \ll q\sqrt{pp'} \sum_{\substack{0<|s|\le \frac{pp'q}{2} \\ anpp'+s\equiv 0 \pmod q}} |s|^{-1}\,. \tag{36}$$

From (29), (31) and (36) we obtain

$$\begin{aligned} \mathcal{E}_2^{(1)} &\ll N^{\varepsilon}\, q\, P \sum_{P<p,p'\le 2P} \sum_{n\le M^{1+\varepsilon}} \sum_{\substack{0<|s|\le 2P^2 q \\ anpp'+s\equiv 0 \pmod q}} |s|^{-1} \\ &\ll N^{\varepsilon}\, q\, P \sum_{0<|s|\le 2P^2 q} |s|^{-1} \sum_{\substack{h\le 4P^2 M^{1+\varepsilon} \\ ah+s\equiv 0 \pmod q}} \tau^2(h)\,. \end{aligned}$$

Obviously, the inner sum is $\ll N^{\varepsilon}(1 + P^2 N D^{-2} q^{-1})$ and using (1) and (3) we get

$$\mathcal{E}_2^{(1)} \ll N^{\varepsilon}\, P \left(q + \frac{P^2 N}{D^2}\right) \ll N^{1+\varepsilon} P \Big(\frac{1}{Q} + \frac{P^2}{D^2}\Big)\,. \tag{37}$$

From (17), (18), (25), (27), (28) and (37) we find that

$$Y(\alpha, D) \ll N^{1+\varepsilon} \Big(\frac{1}{PQ} + \frac{P}{D^2} + \frac{D^2}{PN}\Big)\,. \tag{38}$$

We choose $P = N^{\varepsilon - 1/2} D^2$. It follows from (1) and (7) that the condition (8) holds. From (38) we obtain

$$Y(\alpha, D) \ll N^{1+\varepsilon} \Big(\frac{1}{PQ} + \frac{1}{N^{1/2}}\Big) \ll N^{1+\varepsilon} Q^{-1}\,.$$

It remains to apply (4) and the proof of the theorem is complete.

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Department of Mathematics
Plovdiv University "P. Hilendarski"
24 "Tsar Asen" str.
Plovdiv 4000
Bulgaria
Email: dtolev@pu.acad.bg