On the distribution of r-tuples of squarefree numbers in short intervals

D. I. Tolev

Abstract

We consider the number of r-tuples of squarefree numbers in a short interval. We prove that it cannot be much bigger than the expected value and we also establish an asymptotic formula if the interval is not very short.

Mathematics Subject Classification (2000): 11N25, 11N36, 11N37.

Keywords: squarefree numbers, sieve methods.

1 Introduction and statement of the results.

For $x \geq 1$ and $h \geq 1$ we define

$$Q(x) = \sum_{n \leq x} \mu^2(n), \qquad Q(x,h) = Q(x+h) - Q(x),$$

where $\mu(n)$ is the Möbius function. It is conjectured that for any $\varepsilon > 0$ there exists $x_0(\varepsilon)$ such that $Q(x, x^{\varepsilon}) > 0$ whenever $x \geq x_0(\varepsilon)$. A conditional proof of this hypothesis (under the ABC-conjecture) was found in 1998 by Granville [3]. An unconditional proof is not known at present, but many approximations were established during the last decades. The strongest of them is due to Filaseta and Trifonov [2] (an information about the earlier work on this problem is also available there). In 1992, using clever elementary arguments, they proved the following:

Theorem 1 (Filaseta, Trifonov). *There exists a constant* $c > 0$ *such that if* x *is sufficiently large and* $h = c\,x^{1/5} \log x$, *then* $Q(x,h) > 0$.

In [2] it is established, actually, that under the hypotheses of Theorem 1 one has $Q(x,h) \geq c_1 h$ for some $c_1 > 0$. As we shall see later, using a slight modification of the method of Filaseta and Trifonov, we can prove that $Q(x,h) \sim \frac{6}{\pi^2} h$ when $x \to \infty$ and $\frac{h}{x^{1/5} \log x} \to \infty$. It is not known whether $Q(x,h) > 0$ for smaller h, but in this case we establish that $Q(x,h)$ cannot be much bigger than $\frac{6}{\pi^2} h$.

In the present paper we find results of this type for a more general problem. Let $\mathbf{l} = \langle l_1, \dots, l_r \rangle$ be a vector with distinct, non-negative, integer components and define

$$Q_{\mathbf{l}}(x) = \sum_{n \leq x} \mu^2(n+l_1) \dots \mu^2(n+l_r), \qquad Q_{\mathbf{l}}(x,h) = Q_{\mathbf{l}}(x+h) - Q_{\mathbf{l}}(x).$$

An asymptotic formula for $Q_{\mathbf{l}}(x)$ in the case $\mathbf{l} = \langle 0, 1 \rangle$ was established in an elementary way by Carlitz [1]. Hall [5] found an asymptotic formula in the general case and also proved some results concerning the behavior of $Q_{\mathbf{l}}(x,h)$ on average. Later Heath-Brown [6] considered again the particular case $\mathbf{l} = \langle 0, 1 \rangle$ and, using his square sieve, improved the estimate of the error term in the asymptotic formula for $Q_{\mathbf{l}}(x)$. Finally, Tsang [7] applied the Buchstab–Rosser sieve as well as Heath-Brown's method and proved that if $r \leq \frac{1}{25}(\log x / \log\log x)$ and $l_1, \dots, l_r \leq cx$ for some constant $c > 0$, then

$$(1) \quad A(\mathbf{l})\, x + O\left(r^{12/5} x^{3/5} (\log x)^{-8/5}\right) \leq Q_{\mathbf{l}}(x) \leq A(\mathbf{l})\, x + O\left(r^2 x^{7/11} (\log x)^7\right),$$

where the constants in the O-terms depend only on c,

$$(2) \qquad A(\mathbf{l}) = \prod_p \left(1 - \frac{u(p)}{p^2}\right)$$

(the product is taken over all primes p) and where $u(p)$ is the number of distinct residue classes modulo p^2 represented by the integers $l_1, \dots, l_r$.

Our main result is the following theorem, which states that the number of r-tuples of squarefree integers, lying even in a very short interval, cannot be much bigger than the expected value.

Theorem 2. *Let x, h be real numbers such that $10^3 \leq h \leq x$ and let $r, l_1, \dots, l_r$ be integers satisfying*

$$(3) \qquad 1 \leq r \leq \frac{\log h}{\log\log h}, \qquad 0 \leq l_1 < \dots < l_r \leq x.$$

Then we have

$$Q_{\mathbf{l}}(x,h) \leq A(\mathbf{l})\, h \left\{1 + O\left(h^{-1/3+\rho(h)}\right)\right\}, \qquad \rho(h) = 2\frac{\log\log\log h}{\log\log h}. \tag{4}$$

where the constant in the O-term is absolute.

For the proof we apply Selberg's sieve. We note that the upper bound for $Q_{\mathbf{l}}(x,h)$, given by (4), does not depend on x.

Our second result is a generalization of Theorem 1. We apply again sieve methods (Buchstab's identity and the sieve of Eratosthenes) as well as a version of the main proposition of [2] and prove the following:

Theorem 3. *Let x be sufficiently large and $\psi(x)$ be a monotonically increasing function, such that*

$$2 \leq \psi(x) \leq e^{-10}(\log x)^{2/3}. \tag{5}$$

Suppose that the integers $r, l_1, \ldots, l_r$ satisfy

$$1 \leq r, \qquad e^{10\sqrt{r}} \leq (\log x)^{2/3}\psi(x)^{-1}, \qquad 0 \leq l_1 < \cdots < l_r \leq x \tag{6}$$

and let

$$h \geq e^{10\sqrt{r}}\, \psi(x)\, x^{1/5} \log x. \tag{7}$$

Then we have

$$Q_{\mathbf{l}}(x,h) = A(\mathbf{l})\, h \left\{1 + O\left(\psi(x)^{-1}\right)\right\}, \tag{8}$$

where the constant in the O-term is absolute.

We note that a week version of Theorem 2 can be deduced from the proof of Theorem 3 (see (38), (43) and (44)). More precisely, if the conditions (5) and (6) hold and if $x^{\varepsilon} < h \leq x$, where $\varepsilon > 0$ is arbitrarily small, then

$$Q_{\mathbf{l}}(x,h) \leq A(\mathbf{l})\, h \left\{1 + O\left(\psi(h)^{-1}\right)\right\}$$

(the constant in the O-term depends on ε). However the estimate of the remainder term in (4) is much sharper and also in Theorem 2 we do not impose a lower bound for h depending on x.

Acknowledgement: The author is grateful to Professor D.R.Heath-Brown for reading the preliminary version of the paper and for some useful suggestions and remarks. The author also thanks the Ministry of Science and Education of Bulgaria for the financial support under grant MI-1504/2005.

2 Notations, lemmas and some simple estimates.

As usual, $\mu(n)$ is the Möbius function and $\omega(n)$ denotes the number of distinct prime factors of n. The letters p and q are reserved for prime numbers. By (k_1, k_2) and $[k_1, k_2]$ we denote the greatest common divisor and, respectively, the least common multiple of the integers k_1 and k_2. In this way we also denote open and, respectively, closed intervals, but the meaning is always clear from the context. We write $\#\mathcal{M}$ for the cardinality of the finite set $\mathcal{M}$. If it is not specified explicitly, the constants in the O-terms and $\ll$-symbols are absolute.

For a positive integer k we define

$$\sigma(k) = \prod_{p^2 | k} p.$$

If n is a positive integer and if $\mathbf{l} = \langle l_1, \ldots, l_r \rangle$ is a vector with non-negative integer components we define

$$\xi(n) = \xi_{\mathbf{l}}(n) = \prod_{j=1}^{r} \sigma(n + l_j).$$

For any real $z \geq 2$ we denote

$$P(z) = \prod_{p<z} p.$$

We write $\mathcal{D}(z, k)$ for an abbreviation of the condition $(\sigma(k), P(z)) = 1$. If $z_1, z_2, \ldots, z_r \geq 2$ then we introduce another condition $\mathcal{E}_{n,\mathbf{l}}(z_1, z_2, \ldots, z_r)$, which means that n satisfies $\mathcal{D}(z_i, n + l_i)$ for all $i = 1, \ldots, r$.

We shall see that under the conditions

$$1 \leq h \leq x, \qquad 0 \leq l_1, \ldots, l_r \leq x \tag{9}$$

we have

$$Q_{\mathbf{l}}(x, h) = \#\{n \in (x, x+h] \ : \ \mathcal{E}_{n,\mathbf{l}}(2\sqrt{x},\, 2\sqrt{x}, \ldots, 2\sqrt{x}\,) \,\}. \tag{10}$$

Indeed, using (9), we find that $n + l_i < 4x$ for any $n \in (x, x+h]$ and for $i = 1, \ldots, r$. Therefore the integer $n + l_i$ is squarefree if and only if the condition $\mathcal{D}(2\sqrt{x}, n + l_i)$ holds. This implies the representation (10).

For any squarefree integer d we define

$$u(d) = \prod_{p|d} u(p) \tag{11}$$

and

$$N_d(x, h) = \#\{n \in (x, x+h] \; : \; \xi(n) \equiv 0 \pmod{d} \}. \tag{12}$$

As it is mentioned in [7], p. 269, the congruence $\xi(n) \equiv 0 \pmod{p}$ has $u(p)$ solutions modulo p^2 and, respectively, the congruence $\xi(n) \equiv 0 \pmod{d}$, where d is squarefree, has exactly $u(d)$ solutions modulo d^2. Therefore we find

$$N_d(x, h) = h\frac{u(d)}{d^2} + O(u(d)). \tag{13}$$

Obviously, for any prime p we have

$$u(p) \le r. \tag{14}$$

We can also assume that

$$u(p) \le p^2 - 1 \tag{15}$$

for all p because otherwise we would have $Q_{\mathbf{l}}(x, h) = A(\mathbf{l}) = 0$ and our results would be trivial.

We shall use the following simple estimate:

$$1 \le A(\mathbf{l})^{-1} \le e^{9\sqrt{r}}. \tag{16}$$

Indeed, the first inequality is obvious. To prove the second one we use (2) to write

$$A(\mathbf{l})^{-1} = \prod_{p \le \sqrt{2r}} \left(1 - \frac{u(p)}{p^2}\right)^{-1} \prod_{p > \sqrt{2r}} \left(1 - \frac{u(p)}{p^2}\right)^{-1} = P_1 P_2, \tag{17}$$

say. From (15) and from the well-known upper bound in the Tchebyshev prime number theorem

$$\prod_{p\le w} p \le 4^w \qquad \text{for} \qquad w \ge 1$$

we find

$$P_1 \le \prod_{p\le\sqrt{2r}} \left(1 - \frac{p^2-1}{p^2}\right)^{-1} = \prod_{p\le\sqrt{2r}} p^2 \le 4^{2\sqrt{2r}}. \tag{18}$$

Respectively, from (14) we get

$$\log P_2 \le - \sum_{p>\sqrt{2r}} \log(1 - rp^{-2}) \le 2r \sum_{n>\sqrt{2r}} n^{-2} \le 2\sqrt{2r}. \tag{19}$$

The right inequality in (16) follows from (17) – (19).

The core of the proof of Filaseta and Trifonov's theorem is the following:

Lemma 1 (Filaseta, Trifonov)**.** *Suppose that x is sufficiently large and $h = cx^{1/5}\log x$, where $c > 1$ is a sufficiently large constant. Then there exists a constant $\gamma > 0$ such that*

$$\#\{d \in (\,h\sqrt{\log x}, 2\sqrt{x}\,] \cap \mathbb{Z} \;:\; md^2 \in (x, x+h] \;\; \textit{for some} \;\; m \in \mathbb{Z}\,\} \ll c^{-\gamma} h.$$

The proof of this result is presented in detail in [2]. To establish our Theorem 3 we shall use the following modification:

Lemma 2. *Suppose that x is sufficiently large and*

$$x \le X \le 2x, \qquad 1 \le R \le (\log x)^{2/3},$$
$$h \ge R\,x^{1/5}\log x, \qquad \lambda = R^{-1}\,h\,\log x \le 2\sqrt{x}.$$

Then we have

$$\#\{d \in [\,\lambda, 2\sqrt{x}\,] \cap \mathbb{Z} \;:\; md^2 \in (X, X+h] \;\; \textit{for some} \;\; m \in \mathbb{Z}\,\} \;\ll\; R^{-1}h.$$

The proof differs only very slightly from the proof of Lemma 1, so we omit it.

3 Proof of Theorem 2.

We may assume that $h \geq h_0$, where h_0 is a sufficiently large absolute constant. We use the representation of $Q_{\mathfrak{l}}(x,h)$ in the form (10) and, since the condition $\mathcal{E}_{n,\mathfrak{l}}(\,2\sqrt{x},\,2\sqrt{x},\ldots,2\sqrt{x}\,)$ is equivalent to $(\xi(n),P(2\sqrt{x}))=1$, we can write

$$Q_{\mathfrak{l}}(x,h)=\sum_{x<n\leq x+h}\ \sum_{d|(\xi(n),P(2\sqrt{x}))}\mu(d).$$

Now we apply Selberg's upper bound sieve. Let $\lambda(d)$ be real numbers defined for squarefree integers d. We suppose that $\lambda(1)=1$ and $\lambda(d)=0$ for $d>z$, where z is a parameter for which we assume

$$2<z<2\sqrt{x}. \tag{20}$$

We note that if d is squarefree and $d\leq z$ then $d \mid P(2\sqrt{x})$. Hence we find

$$Q_{\mathfrak{l}}(x,h)\leq\sum_{x<n\leq x+h}\left(\sum_{d|\xi(n)}\lambda(d)\right)^2=\sum_{x<n\leq x+h}\ \sum_{d_1,d_2|\xi(n)}\lambda(d_1)\lambda(d_2) \tag{21}$$

$$=\sum_{d_1,d_2\leq z}\lambda(d_1)\lambda(d_2)\ N_{[d_1,d_2]}(x,h),$$

where $N_d(x,h)$ is defined by (12).

We use (13) and (21) to get

$$Q_{\mathfrak{l}}(x,h)\leq hV+O(R\,), \tag{22}$$

where

$$V=\sum_{d_1,d_2\leq z}\frac{\lambda(d_1)\lambda(d_2)}{[d_1,d_2]^2}\,u([d_1,d_2]),\quad R=\sum_{d_1,d_2\leq z}|\lambda(d_1)|\,|\lambda(d_2)|\,u([d_1,d_2]). \tag{23}$$

We define $\lambda(d)$ for $1<d\leq z$ in such a way as to minimize V. By a straightforward application of Selberg's method (see, for example, [4], Chapter 3) we can verify that the optimal choice is

$$\lambda(d)=\mu(d)\prod_{p|d}\left(1-\frac{u(p)}{p^2}\right)^{-1}\frac{H(z/d,d)}{H(z)}, \tag{24}$$

where

$$(25)\quad H(y,m)=\sum_{\substack{k\le y\\(k,m)=1}}\frac{\mu^2(k)\,u(k)}{k^2}\prod_{p|k}\left(1-\frac{u(p)}{p^2}\right)^{-1},\qquad H(y)=H(y,1).$$

In this case, one can obtain that the minimal value of V is

$$(26)\qquad V_{\min}=H(z)^{-1}.$$

From (2), (24) and (25) we easily find

$$(27)\qquad |\lambda(d)|\le\mu^2(d)\prod_{p|d}\left(1-\frac{u(p)}{p^2}\right)^{-1}\le A(\mathbf{1})^{-1}\mu^2(d).$$

Having in mind (22), (23) and (26) we obtain

$$(28)\qquad Q_1(x,h)\le hH(z)^{-1}+O(G^2),$$

where

$$G=\sum_{d\le z}|\lambda(d)|u(d).$$

From (11), (14) and (27) we find

$$(29)\qquad G\le A(\mathbf{1})^{-1}\sum_{d\le z}r^{\omega(d)}\mu^2(d)=A(\mathbf{1})^{-1}U,$$

say. To estimate the sum U we assume that

$$(30)\qquad \nu=1+\frac{r}{\log z}\le 2$$

and, using Euler's identity and the elementary properties of Riemann's zeta-function, we easily get

$$(31)\qquad \begin{aligned}U&\le\sum_{d=1}^{\infty}\left(\frac{z}{d}\right)^{\nu}\mu^2(d)\,r^{\omega(d)}=z^{\nu}\prod_{p}\left(1+\frac{r}{p^{\nu}}\right)\le z^{\nu}\left(\frac{\nu}{\nu-1}\right)^{r}\\&\le z\,(2\,e\,r^{-1}\log z)^{r}.\end{aligned}$$

Applying Euler's identity we also find

$$\sum_{k=1}^{\infty} \frac{\mu^2(k)\, u(k)}{k^2} \prod_{p|k} \left(1 - \frac{u(p)}{p^2}\right)^{-1} = A(\mathbf{l})^{-1}.$$

Hence, if we define $\omega(z)$ by

$$H(z) = A(\mathbf{l})^{-1} - \omega(z), \tag{32}$$

then, using (2), (11) and (14) we get

$$\begin{aligned} 0 \leq \omega(z) &= \sum_{k>z} \frac{\mu^2(k) u(k)}{k^2} \prod_{p|k} \left(1 - \frac{u(p)}{p^2}\right)^{-1} \\ &\leq A(\mathbf{l})^{-1} \sum_{k>z} \frac{\mu^2(k)\, r^{\omega(k)}}{k^2} = A(\mathbf{l})^{-1} U_1, \end{aligned} \tag{33}$$

say. Arguing as in the proof of (31) we find

$$U_1 \ll z^{-1}(2\, e\, r^{-1} \log z)^r.$$

Using this last estimate, (32) and (33) we establish

$$H(z)^{-1} = A(\mathbf{l}) \left\{1 + O(z^{-1}(2\, e\, r^{-1} \log z)^r)\right\}. \tag{34}$$

From (28), (29), (31) and (34) we obtain

$$Q_{\mathbf{l}}(x, h) \leq A(\mathbf{l})\, h \left\{1 + O(\Delta)\right\}, \tag{35}$$

where

$$\Delta = z^{-1}(2e\, r^{-1} \log z)^r + A(\mathbf{l})^{-3} h^{-1} z^2 (2e\, r^{-1} \log z)^{2r}.$$

We choose

$$z = h^{1/3}\, (r^{-1} \log h)^{-r/3}.$$

and note that the conditions (20) and (30) are satisfied. Using (3) and (16) we find

$$\Delta \ll h^{-1/3+\rho(h)},$$

where $\rho(h)$ is specified by (4).

From the last formula and (35) we obtain (4), thus completing the proof of Theorem 2.

4 Proof of Theorem 3.

We can assume that $h \le x^{7/11}(\log x)^{10}$ because otherwise (8) is a consequence of (1), (6) and (16). Suppose that $\mathcal{A} \subset (x, x+h] \cap \mathbb{Z}$ and let

$$\lambda_0 = e^{10\sqrt{r}}\, \psi(x). \tag{36}$$

Using the Buchstab identity (see, for example, [4], Chapter 7, p. 204), we find for any $i = 1, \dots, r$ that

$$\begin{aligned} \#\{n \in \mathcal{A} \ :\ \mathcal{D}(2\sqrt{x}, n + l_i)\ \} \\ &= \#\{n \in \mathcal{A} \ :\ \mathcal{D}(\lambda_0, n + l_i)\ \} \\ &- \sum_{\lambda_0 \le q < 2\sqrt{x}} \#\{n \in \mathcal{A} \ :\ q^2 \mid n + l_i\ ,\ \mathcal{D}(q, n + l_i)\ \}. \end{aligned} \tag{37}$$

Having written $Q_{\mathbf{l}}(x,h)$ in the form (10), we apply (37) with $i = 1$ and with the set $\mathcal{A}$ consisting of the integers $n \in (x, x+h]$ satisfying the conditions $\mathcal{D}(2\sqrt{x}, n + l_j)$ for all $j = 2, \dots, r$. We get

$$\begin{aligned} Q_{\mathbf{l}}(x,h) &= \#\{n \in (x, x+h] \ :\ \mathcal{E}_{n,\mathbf{l}}(\lambda_0, 2\sqrt{x}, \dots, 2\sqrt{x}\,)\ \} \\ &\quad - \sum_{\lambda_0 \le q < 2\sqrt{x}} \#\{n \in (x, x+h] \ :\ q^2 \mid n + l_1\ ,\ \mathcal{E}_{n,\mathbf{l}}(q, 2\sqrt{x}, \dots, 2\sqrt{x}\,)\ \}. \end{aligned}$$

We consider the first term from the right side of the last identity and apply (37) again, this time with $i = 2$ and with the set $\mathcal{A}$ consisting of all integers $n \in (x, x+h]$, which satisfy $\mathcal{D}(\lambda_0, n + l_1)$ and $\mathcal{D}(2\sqrt{x}, n + l_j)$ for all $j = 3, \dots, r$. In this way we find another identity for $Q_{\mathbf{l}}(x,h)$.

Proceeding in this manner we obtain

$$Q_{\mathbf{l}}(x,h) = R_0 - \sum_{\nu=1}^{r} \sum_{\lambda_0 \le q < 2\sqrt{x}} R_{\nu,q} = R_0 - \Sigma, \tag{38}$$

say, where

$$\begin{aligned} R_0 &= \#\{n \in (x, x+h] \ :\ \mathcal{E}_{n,\mathbf{l}}(\lambda_0, \lambda_0, \dots, \lambda_0)\ \}, \\ R_{\nu,q} &= \#\big\{n \in (x, x+h] \ :\ q^2 \mid n + l_\nu\ ,\ \mathcal{E}_{n,\mathbf{l}}\big(\underbrace{\lambda_0, \dots, \lambda_0}_{\nu-1},\ q,\ \underbrace{2\sqrt{x}, \dots, 2\sqrt{x}}_{r-\nu}\ \big)\ \big\}. \end{aligned}$$

First consider R_0. It is clear that the condition $\mathcal{E}_{n,\mathbf{l}}(\lambda_0, \lambda_0, \ldots, \lambda_0)$ is equivalent to $(\xi(n), P(\lambda_0)) = 1$. Therefore

$$R_0 = \sum_{x<n\le x+h} \sum_{d|(\xi(n),P(\lambda_0))} \mu(d) = \sum_{d|P(\lambda_0)} \mu(d) N_d(x,h), \tag{39}$$

where $N_d(x,h)$ is defined by (12). Using (13) and (39) we get

$$R_0 = hW + O(H), \tag{40}$$

where

$$W = \sum_{d|P(\lambda_0)} \mu(d) \frac{u(d)}{d^2} = \prod_{p<\lambda_0} \left(1 - \frac{u(p)}{p^2}\right), \qquad H = \sum_{d|P(\lambda_0)} u(d).$$

Consider W and H. Arguing as in the proof of (16) and using (2), (14) and (36) we find

$$W = \prod_p \left(1 - \frac{u(p)}{p^2}\right) \prod_{p\ge\lambda_0} \left(1 - \frac{u(p)}{p^2}\right)^{-1} = A(\mathbf{l}) \left(1 + O\left(\frac{r}{\lambda_0}\right)\right). \tag{41}$$

Respectively, we have

$$H \le \sum_{d|P(\lambda_0)} r^{\omega(d)} = \prod_{p<\lambda_0} (1+r) < (1+r)^{\lambda_0}. \tag{42}$$

From (40) – (42) we get

$$R_0 = A(\mathbf{l})\, h\, \left(1 + O(\Delta)\right),$$

where

$$\Delta = r\lambda_0^{-1} + h^{-1} A(\mathbf{l})^{-1} (1+r)^{\lambda_0}.$$

Ii is not difficult to verify, using (6), (7), (16) and (36), that $\Delta \ll \psi(x)^{-1}$ and we find

$$R_0 = A(\mathbf{l})\, h\, \left(1 + O(\psi(x)^{-1})\right). \tag{43}$$

Consider now the sum Σ, specified by (38). We have

$$0 \le \Sigma \le r \max_{1\le\nu\le r} S_\nu, \tag{44}$$

where

$$S_\nu = \sum_{\lambda_0 \le q < 2\sqrt{x}} \#\{ n \in (x, x+h] \ : \ q^2 \mid n + l_\nu \ \}.$$

We shall prove that

$$S_\nu \ll e^{-10\sqrt{r}}\, \psi(x)^{-1}\, h. \tag{45}$$

Define

$$\lambda = e^{-10\sqrt{r}}\, \psi(x)^{-1}\, h\, \log x. \tag{46}$$

First we assume that $\lambda < 2\sqrt{x}$. In this case we divide S_ν into two parts

$$S_\nu = S' + S''. \tag{47}$$

In S' the summation is taken over the primes $q \in [\lambda_0, \lambda)$ and S'' is the contribution from the primes $q \in [\lambda, 2\sqrt{x})$.

Using Tchebyshev's prime number theorem and (6), (7), (36) and (46) we get

$$S' = \sum_{\lambda_0 \le q < \lambda} \sum_{\substack{x < n \le x+h \\ n + l_\nu \equiv 0 \pmod{q^2}}} 1 \le \sum_{\lambda_0 \le q < \lambda} \left(\frac{h}{q^2} + 1 \right) \ll \frac{h}{\lambda_0} + \frac{\lambda}{\log \lambda} \tag{48}$$

$$\ll e^{-10\sqrt{r}}\, \psi(x)^{-1}\, h.$$

To estimate S'' we write it in the form

$$S'' = \sum_{\lambda \le q < 2\sqrt{x}} \#\{ m \in (x + l_\nu, x + l_\nu + h] \ : \ q^2 \mid m \ \}$$

and apply Lemma 2 with $X = x + l_\nu$ and $R = e^{10\sqrt{r}}\psi(x)$. We find

$$S'' \ll e^{-10\sqrt{r}}\, \psi(x)^{-1}\, h. \tag{49}$$

The estimate (45) follows from (47) – (49).

In the case $\lambda \ge 2\sqrt{x}$ we proceed as in the estimation of S' and easily find that (45) holds as well.

From (16), (38), (43) – (45) and (47) we obtain (8) and Theorem 3 is proved.

References

[1] L. Carlitz, *On a problem in additive arithmetic*, Quart. J. Math. (Oxford), 3, (1932), 273–290.

[2] M. Filaseta and O. Trifonov, *On gaps between squarefree numbers II*, J. London Math. Soc. (2) 45, (1992), 215–221.

[3] A. Granville, *ABC allows us to count squarefrees*, Internat. Math. Res. Notices, 19, (1998), 991–1009.

[4] H. Halberstam and H. E. Richert, *Sieve methods*, Academic Press, 1974.

[5] R. R. Hall, *Squarefree numbers on short intervals*, Mathematika, 29, (1982), 7–17.

[6] D. R. Heath-Brown, *The square sieve and consecutive square-free numbers*, Math. Ann. 266, (1984), 251–259.

[7] K.-M. Tsang, *The distribution of r-tuples of square-free numbers*, Mathematika, 32, (1985), 265–275.

Department of Mathematics
Plovdiv University “P. Hilendarski”
24 “Tsar Asen” str.
Plovdiv 4000
Bulgaria
Email: dtolev@pu.acad.bg