ON THE DISTRIBUTION OF αp MODULO ONE FOR PRIMES p OF A SPECIAL FORM

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Abstract. A classical problem in analytic number theory is to study the distribution of αp modulo 1, where α is irrational and p runs over the set of primes. We consider the subsequence generated by the primes p such that $p+2$ is an almost-prime (the existence of infinitely many such p is another topical result in prime number theory) and prove that its distribution has a similar property.

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1. Introduction and statements of the result

The famous prime twins conjecture states that there exist infinitely many primes p such that $p+2$ is a prime too. This hypothesis is still unproved but there are many approximation to it established. One of the most interesting of them is due to Chen [1]. In 1973 he proved that there are infinitely many primes p for which $p+2 = \mathrm{P}_2$. (As usual P_r denotes an integer with no more than r prime factors, counted according to multiplicity).

Suppose that we have a problem including primes and let $r \geq 2$ be an integer. Having in mind Chen's result we may consider this problem with primes p, such that $p+2 = \mathrm{P}_r$. We will give several examples.

In 1937, Vinogradov [16] proved that every sufficiently large odd n can be represented in the form

$$p_1 + p_2 + p_3 = n, \tag{1}$$

where p_1, p_2, p_3 are primes. In 2000 Peneva [12] and Tolev [13] considered (1) with primes of the form specified above. It was established in [13] that if n is sufficiently large and $n \equiv 3 \pmod 6$ then (1) has a solution in primes p_1, p_2, p_3 such that

$$p_1 + 2 = \mathrm{P}_2, \quad p_2 + 2 = \mathrm{P}_5, \quad p_3 + 2 = \mathrm{P}_7.$$

Further, in 1938 Hua [8] proved that every sufficiently large $n \equiv 5 \pmod{24}$ can be represented as

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 = n, \tag{2}$$

where $p_1, \dots, p_5$ are primes. In 2000 Tolev [15] proved that every sufficiently large $n \equiv 5 \pmod{24}$ can be represented in the form (2) with primes $p_1, \dots, p_5$ such that

$$p_1 + 2 = P_2,\ p_2 + 2 = P_2',\ p_3 + 2 = P_5,\ p_4 + 2 = P_5',\ p_5 + 2 = P_7.$$

Finally, in 2004 Green and Tao [3] proved their celebrated theorem stating that for every natural $k \geq 3$ there are infinitely many arithmetical progression of k different primes. Later they established (see [4]) that there exist infinitely many arithmetical progression of three different primes p, such that $p + 2 = \mathrm{P}_2$. (A weaker result of this type was previously obtained by Tolev [14]). In the paper [4] Green and Tao state that using the same method their result can be extended for progression of k terms, where k is arbitrary large.

In the present paper we consider another popular problem with primes and study it with primes of the form specified above.

Let α be irrational real number, β be real and let $||x|| = \min_{n \in \mathbb{Z}} |x - n|$. In 1947 Vinogradov [17] proved that if $0 < \theta < 1/5$ then there are infinitely many primes p such that

$$||\alpha p + \beta|| < p^{-\theta}.$$

Latter the upper bound for θ was improved and the strongest published result is due to Heath-Brown and Jia [7] with $\theta < 16/49$. We shall prove the following

Theorem 1. *Let $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, $\beta \in \mathbb{R}$ and let $0 < \theta \leq 1/100$. Then there are infinitely many primes p satisfying $p + 2 = \mathrm{P}_4$ and such that*

$$||\alpha p + \beta|| < p^{-\theta}. \tag{3}$$

Other versions of this theorem are also possible, but our intention is to present here a result with r as small as possible and for this r to find some (not necessarily the biggest possible) θ.

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2. Notation

Let N be a sufficiently large real number and δ, η, θ, ρ, κ be real constants satisfying

$$0 < \theta < \eta < \frac{\delta}{2} < \frac{1}{4}, \qquad \eta < \rho < \delta, \qquad 0 < \kappa, \qquad 0 < \theta \leq \frac{1}{100}. \tag{4}$$

We shall specify δ, η, ρ and κ latter. We put

$$\begin{aligned} z &= N^{\eta}, \quad y = N^{\rho}, \quad D = N^{\delta}, \\ \Delta &= \Delta(N) = N^{-\theta}, \quad H = \Delta^{-1} \log^2 N. \end{aligned} \tag{5}$$

By p and q we always denote primes. As usual $\Omega(n)$, $\varphi(n)$, $\mu(n)$, $\Lambda(n)$ denote respectively the numbers of prime factors of n counted with the multiplicity, Euler's function, Möbius' function and Mangoldt's function. We denote by $\tau_k(n)$ the number of solutions of the equation $m_1 m_2 \dots m_k = n$ in natural numbers $m_1, \dots, m_k$ and $\tau_2(n) = \tau(n)$. Let $(m_1, \dots, m_k)$ and $[m_1, \dots, m_k]$ be the greatest common divisor and the least common multiple of $m_1, \dots, m_k$ respectively. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n(k)$. As usual, $[y]$ denotes the integer part of y, $||y||$ – the distance from y to the nearest integer, $e(y) = e^{2\pi i y}$. For positive A and B we write $A \asymp B$ instead of $A \ll B \ll A$. The letter ε denotes an arbitrary small positive number, not the same in all appearances. For example this convention allows us to write $x^\varepsilon \log x \ll x^\varepsilon$.

3. Proof of the theorem

We take a periodic with period 1 function such that

$$\begin{aligned} 0 < \chi(t) < 1 \quad &\text{if} \quad -\Delta < t < \Delta; \\ \chi(t) = 0 \quad &\text{if} \quad \Delta \le t \le 1 - \Delta, \end{aligned} \tag{6}$$

and which has a Fourier series

$$\chi(t) = \Delta + \sum_{|k|>0} g(k) e(kt), \tag{7}$$

with coefficients satisfying

$$\begin{aligned} g(0) &= \Delta, \\ g(k) &\ll \Delta \qquad \text{for all } k, \\ \sum_{|k|>H} |g(k)| &\ll N^{-1}. \end{aligned} \tag{8}$$

The existence of such a function is a consequence of a well known lemma of Vinogradov (see [11], ch. 1, §2).

Consider the sum

$$\Gamma = \Gamma(N) = \sum_{\substack{N/2 < p \le N \\ (p+2,\, P(z))=1}} \chi(\alpha p + \beta)\, T_p \log p, \tag{9}$$

where

$$P(z) = \prod_{2 < p \le z} p \tag{10}$$

and

$$T_p = 1 - \kappa \sum_{\substack{z < q \le y \\ q | p+2}} \left(1 - \frac{\log q}{\log y}\right). \tag{11}$$

Obviously

$$\Gamma(N) \le \Gamma_1 \tag{12}$$

where Γ_1 is the sum of the terms of $\Gamma(N)$ for which $T_p > 0$. Denote by Γ_2 the sum of the term of Γ_1 for which $\mu^2(p+2) = 0$. It is clear that

$$0 \le \Gamma_2 \ll \sum_{z \le q} \sum_{\substack{n \le N \\ n+2 \equiv 0 (q^2)}} \log n \ll \log N \sum_{z \le q \le \sqrt{N+2}} \left(\frac{N}{q^2} + 1 \right) \tag{13}$$

$$\ll \frac{N^{1+\varepsilon}}{z} + N^{\frac{1}{2}+\varepsilon} \ll N^{1-\eta+\varepsilon}.$$

We also remove from Γ_1 the term (if such exist) for which $N-2 < p \le N$ and the resulting error is $O(\log N)$. Therefore

$$\Gamma \le \Gamma_3 + O(N^{1-\eta+\varepsilon}), \tag{14}$$

where

$$\Gamma_3 = \sum \chi(\alpha p + \beta) \, T_p \, \log p$$

and where the summation is taken over the primes p, satisfying

$$N/2 < p \le N - 2, \tag{15}$$

$$T_p > 0, \quad \mu^2(p+2) = 1, \quad (p+2, P(z)) = 1. \tag{16}$$

Assume that

$$\Gamma(N) \gg \frac{\Delta N}{\log N}. \tag{17}$$

Then from (4), (5) and (14) we get $\Gamma_3 > 0$. Hence there exist a prime p satisfying (15), (16) and such that

$$\chi(\alpha p + \beta) > 0. \tag{18}$$

From (5), (6), (15) and (18) it follows that this prime satisfies (3).

On the other hand, from the properties of the weights T_p (see [5], ch. 9) it follows that if p satisfies (15), (16) then

$$\Omega(p+2) \le \frac{1}{\kappa} + \frac{1}{\rho}.$$

We see that to prove our theorem it is enough to determine the constants δ, η, θ, ρ, κ in such a way that:

***I*:** There exist a sequence $\{N_j\}_{j=1}^{\infty}$ such that

$$\lim_{j \to \infty} N_j = \infty, \qquad \Gamma(N_j) \gg \frac{\Delta(N_j) \, N_j}{\log N_j}, \quad j = 1, 2, 3, \ldots$$

***II*:** We have

$$\frac{1}{\kappa} + \frac{1}{\rho} < 5 \tag{19}$$

Using (9) and (11) we write Γ as

$$\Gamma = \Phi - \kappa G, \tag{20}$$

where

$$\Phi = \sum_{\substack{N/2<p\le N\\ (p+2,\,P(z))=1}} \chi(\alpha p+\beta)\log p \tag{21}$$

and

$$G = \sum_{\substack{N/2<p\le N\\ (p+2,\,P(z))=1}} \chi(\alpha p+\beta)\log p \sum_{\substack{z<q\le y\\ q\mid p+2}} \left(1-\frac{\log q}{\log y}\right). \tag{22}$$

We shall estimate Φ from below and G from above.

Consider the sum Φ. We apply a lower bound linear sieve. We take the lower Rosser weights $\lambda^-(d)$ of order D. For the definition and their properties we refer the reader to [9], [10]. In particular we shall use that the Rosser weights are real numbers such that

$$|\lambda^-(d)|\le 1, \quad \lambda^-(d)=0 \quad \text{if} \quad d>D \quad \text{or} \quad \mu^2(d)=0, \tag{23}$$

$$\sum_{d\mid A}\lambda^-(d) \le \begin{cases} 1, & \text{if } A=1,\\ 0, & \text{if } A\in\mathbb{N},\ A>1. \end{cases} \tag{24}$$

We shall also use that if

$$s=\frac{\log D}{\log z}=\frac{\delta}{\eta} \qquad \text{and} \qquad 2<s<4 \tag{25}$$

then

$$\sum_{d\mid P(z)}\frac{\lambda^-(d)}{\varphi(d)} \ge \Pi(z)\left(\frac{2e^\gamma\log(s-1)}{s}+O\Big((\log N)^{-1/3}\Big)\right), \tag{26}$$

where

$$\Pi(z)=\prod_{2<p\le z}\left(1-\frac{1}{p-1}\right). \tag{27}$$

From this place onwards we assume that

$$2<\frac{\delta}{\eta}<4, \tag{28}$$

so the inequality (26) holds. Using (21), (24) we get

$$\begin{aligned}\Phi \ge \Phi_1 &= \sum_{N/2<p\le N}\chi(\alpha p+\beta)\log p \sum_{d\mid(p+2,\,P(z))}\lambda^-(d)\\ &= \sum_{d\mid P(z)}\lambda^-(d)\sum_{\substack{N/2<p\le N\\ p+2\equiv 0\,(d)}}\chi(\alpha p+\beta)\log p.\end{aligned} \tag{29}$$

Form (7), (8) we find that

$$\Phi_1=\Delta(\Phi_2+\Phi_3)+O(1), \tag{30}$$

where

$$\Phi_2 = \sum_{d|P(z)} \lambda^-(d) \sum_{\substack{N/2<p\le N\\ p+2\equiv 0\,(d)}} \log p,$$

$$\Phi_3 = \sum_{d|P(z)} \lambda^-(d) \sum_{0<|k|\le H} c(k) \sum_{\substack{N/2<p\le N\\ p+2\equiv 0\,(d)}} e(\alpha pk)\log p, \tag{31}$$

$$c(k) = \Delta^{-1} g(k) e(\beta k) \ll 1. \tag{32}$$

Consider Φ_2. From Bombieri-Vinogradov's theorem (see [2], ch. 24), (4), (5), (23) we see that

$$\Phi_2 = \frac{N}{2}\sum_{d|P(z)} \frac{\lambda^-(d)}{\varphi(d)} + O\bigg(\frac{N}{(\log N)^2}\bigg). \tag{33}$$

It is well known that the product defined by (27) satisfies

$$\Pi(z) \asymp \frac{1}{\log z}. \tag{34}$$

Therefore from (4), (5), (26), (33), (34) we find that

$$\Phi_2 \ge e^{\gamma} N\, \Pi(z)\, \frac{\log(s-1)}{s} + O\bigg(\frac{N}{(\log N)^{4/3}}\bigg), \tag{35}$$

where s is specified by (25). We shall study the sum Φ_3 later.

Consider now the sum G, defined by (22). We write it in the form

$$G = \sum_{z<q<y} \bigg(1 - \frac{\log q}{\log y}\bigg) \sum_{\substack{N/2<p\le N\\ p+2\equiv 0\,(q)\\ (p+2,\,P(z))=1}} \chi(\alpha p+\beta)\log p \tag{36}$$

and then apply an upper bound linear sieve. Let $\lambda_q(d)$ be the upper Rosser weights of order $\dfrac{D}{q}$. We know that

$$|\lambda_q(d)| \le 1, \quad \lambda_q(d) = 0 \quad \text{if} \quad d > \frac{D}{q} \quad \text{or} \quad \mu^2(d) = 0, \tag{37}$$

$$\sum_{d|A} \lambda_q(d) \ge \begin{cases} 1, & \text{if } A = 1,\\ 0, & \text{if } A\in\mathbb{N},\ A>1. \end{cases} \tag{38}$$

We shall also use that if

$$s_1 = \frac{\log(D/q)}{\log z} \quad \text{and} \quad 1 < s_1 < 3 \tag{39}$$

then

$$\sum_{d|P(z)} \frac{\lambda_q(d)}{\varphi(d)} \le \Pi(z)\bigg(\frac{2e^{\gamma}}{s_1} + O((\log N)^{-1/3})\bigg) \tag{40}$$

From this place onwards we assume that

$$\eta + \rho < \delta. \tag{41}$$

Then using also (28) we see that the condition (39) holds, consequently (40) is true. From (36) – (38) we find

$$\begin{aligned} G \le \ G_1 &= \\ &= \sum_{z<q<y} \left(1 - \frac{\log q}{\log y}\right) \sum_{\substack{N/2<p\le N \\ p+2\equiv 0\,(q)}} \chi(\alpha p + \beta) \log p \sum_{d|(p+2,\,P(z))} \lambda_q(d) \\ &= \sum_{z<q<y} \left(1 - \frac{\log q}{\log y}\right) \sum_{d|P(z)} \lambda_q(d) \sum_{\substack{N/2<p\le N \\ p+2\equiv 0\,(qd)}} \chi(\alpha p + \beta) \log p \\ &= \sum_{m\le D} \gamma(m) \sum_{\substack{N/2<p\le N \\ p+2\equiv 0\,(m)}} \chi(\alpha p + \beta) \log p, \end{aligned} \tag{42}$$

where

$$\gamma(m) = \sum_{\substack{z<q<y \\ d|P(z) \\ qd=m}} \left(1 - \frac{\log q}{\log y}\right) \lambda_q(d). \tag{43}$$

Using (10), (37) and (43) we easily see that

$$|\gamma(m)| \le 1. \tag{44}$$

From (7), (8) and (42) we find

$$G_1 = \Delta(G_2 + G_3) + O(1), \tag{45}$$

where

$$\begin{aligned} G_2 &= \sum_{m\le D} \gamma(m) \sum_{\substack{N/2<p\le N \\ p+2\equiv 0\,(m)}} \log p, \\ G_3 &= \sum_{m\le D} \gamma(m) \sum_{0<|k|\le H} c(k) \sum_{\substack{N/2<p\le N \\ p+2\equiv 0\,(m)}} e(\alpha p k) \log p, \end{aligned} \tag{46}$$

and where $c(k)$ satisfies (32).

We apply again Bombieri-Vinogradov's theorem and (4), (5), (44) to find that

$$G_2 = \frac{N}{2} \sum_{m\le D} \frac{\gamma(m)}{\varphi(m)} + O\left(\frac{N}{(\log N)^2}\right). \tag{47}$$

Using (40), (43) we obtain

$$\begin{aligned}(48)\quad \sum_{m\le D}\frac{\gamma(m)}{\varphi(m)} &= \sum_{z<q<y}\left(1-\frac{\log q}{\log y}\right)\sum_{d|P(z)}\frac{\lambda_q(d)}{\varphi(qd)}\\ &= \sum_{z<q<y}\left(1-\frac{\log q}{\log y}\right)\frac{1}{q-1}\sum_{d|P(z)}\frac{\lambda_q(d)}{\varphi(d)}\\ &\le \sum_{z<q<y}\left(1-\frac{\log q}{\log y}\right)\frac{1}{q-1}\\ &\quad\times\Pi(z)\left(2e^{\gamma}\left(\frac{\log(D/q)}{\log z}\right)^{-1}+O\Big((\log N)^{-1/3}\Big)\right).\end{aligned}$$

Therefore from (5), (34), (47), (48) we get
(49)

$$G_2 \le e^{\gamma}N\,\Pi(z)\sum_{z<q<y}\left(1-\frac{\log q}{\log y}\right)\frac{1}{q-1}\left(\frac{\log(D/q)}{\log z}\right)^{-1}+O\left(\frac{N}{(\log N)^{4/3}}\right).$$

Now we are in a position to find a lower bound for the sum Γ. From (20), (29), (30), (35), (42), (45), (49) it follows that

$$(50)\qquad \Gamma \ge e^{\gamma}\Delta\,N\,\Pi(z)\,\Sigma + O\left(\frac{\Delta N}{(\log N)^{4/3}}\right)+O\Big(\Delta|\Phi_3-\kappa G_3|\Big),$$

where

$$(51)\qquad \Sigma = \frac{\log(s-1)}{s} - \kappa\sum_{z<q<y}\left(1-\frac{\log q}{\log y}\right)\frac{1}{q-1}\left(\frac{\log(D/q)}{\log z}\right)^{-1}$$

and where s is specified by (25). Using partial summation and the prime number theorem it is easy to prove that

$$(52)\qquad \Sigma = \Sigma_0 + O\left(\frac{1}{\log N}\right),$$

where

$$(53)\qquad \Sigma_0 = \frac{\log(s-1)}{s} - \kappa\eta\int_{\eta}^{\rho}\left(\frac{1}{u}-\frac{1}{\rho}\right)\frac{1}{\delta-u}\,du.$$

Therefore, using (5), (34), (50) we get

$$\Gamma \ge e^{\gamma}N\,\Delta\,\Pi(z)\,\Sigma_0 + O\left(\frac{\Delta N}{(\log N)^{4/3}}\right)+O\Big(\Delta|\Phi_3-\kappa G_3|\Big).$$

Now we shall see that if N is a term of a suitable sequence tending to infinity then the last error term in the formula above can be omitted. The following lemma holds:

Lemma 1. *Suppose* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ *and*

$$\delta + \theta < \frac{1}{3}\,. \tag{54}$$

Let $\xi(d)$, $c(k)$ *be complex numbers defined for* $d \le D,\ 0 < |k| \le H$, *where* D *and* H *are specified by* (5), *and let*

$$\xi(d) \ll 1, \qquad c(k) \ll 1. \tag{55}$$

Then there exist a sequence $\{N_j\}_{j=1}^{\infty}$, $\lim\limits_{j\to\infty} N_j = \infty$, *such that if*

$$S(N) = \sum_{d\le D} \xi(d) \sum_{1\le |k|\le H} c(k) \sum_{\substack{N/2<p\le N\\ p+2\equiv 0\,(d)}} e(\alpha p k) \log p \tag{56}$$

then we have

$$S(N_j) \ll \frac{N_j}{\log^2 N_j}\,, \qquad j = 1,\, 2,\, 3, \ldots\ .$$

We will present the proof of this Lemma in the next section.

From (31), (46) we see that the quantity $\Phi_3 - \kappa G_3$ can be written as a sum of type (56) with $\xi(d) = \lambda^*(d) - \kappa\gamma(d)$, where $\lambda^*(d) = \lambda^-(d)$ if $d|P(z)$ and $\lambda^*(d) = 0$ otherwise. Using our Lemma we see that there exist a sequence tending to infinity such that if N is its term then

$$\Gamma \ge e^{\gamma} \Delta\, N\, \Pi(z)\, \Sigma_0 + O\biggl(\frac{\Delta N}{(\log N)^{4/3}}\biggr). \tag{57}$$

We put

$$\rho = 0.23, \quad \delta = 0.315, \quad \eta = 0.08, \quad \kappa = 1.58\,.$$

Then it is easy to verify that the conditions (4), (19), (28), (41), (54) are fulfilled and also

$$\Sigma_0 > 0.$$

From the last inequality and (5), (34), (57) it follows that there exist a constant $c > 0$ such that for any N from our sequence we have

$$\Gamma \ge c\,\frac{\Delta N}{\log N} > 0.$$

This completes the proof of the theorem.

4. Proof of Lemma

Since α is irrational we see, using Dirichlet's theorem, that there are infinitely many integers A and natural numbers Q such that

$$\left|\alpha - \frac{A}{Q}\right| < \frac{1}{Q^2}.$$

For any such Q we choose N in a suitable way (see (73)) and in this way define our sequence $\{N_j\}_{j=1}^{\infty}$.

It is clear that

$$S(N) = W + O(HN^{\frac{1}{2}+\varepsilon}), \tag{58}$$

where

$$W = \sum_{N/2<n\le N} \Lambda(n) \sum_{1\le|k|\le H} c(k)e(\alpha nk) \sum_{\substack{d\le D\\ d|n+2\\ 2\nmid d}} \xi(d).$$

Using Heath-Brown's identity [6] with parameters

$$P = N/2,\ P_1 = N,\ u = 2^{-7}N^{\frac{\delta}{2}},\ v = 2^7 N^{\frac{1}{3}},\ w = N^{\frac{1}{2}-\frac{\delta}{4}}. \tag{59}$$

we decompose the sum W as a linear combination of $O(\log^6 N)$ sums of first and second type. The sums of the first type are

$$W_1 = \sum_{M<m\le M_1} a_m \sum_{\substack{L<l\le L_1\\ N/2<ml\le N}} \sum_{0<|k|\le H} c(k)e(\alpha mlk) \sum_{\substack{d\le D\\ d|ml+2\\ 2\nmid d}} \xi(d)$$

and

$$W_1' = \sum_{M<m\le M_1} a_m \sum_{\substack{L<l\le L_1\\ N/2<ml\le N}} \log l \sum_{0<|k|\le H} c(k)e(\alpha mlk) \sum_{\substack{d\le D\\ d|ml+2\\ 2\nmid d}} \xi(d),$$

where

$$M_1 \le 2M, \quad L_1 \le 2L, \quad ML \asymp N, \quad L \ge w, \quad a_m \ll N^{\varepsilon}. \tag{60}$$

The sums of the second type are

$$W_2 = \sum_{M<m\le M_1} a_m \sum_{\substack{L<l\le L_1\\ N/2<ml\le N}} b_l \sum_{0<|k|\le H} c(k)e(\alpha mlk) \sum_{\substack{d\le D\\ d|ml+2\\ 2\nmid d}} \xi(d),$$

where
(61)

$$M_1 \le 2M, \quad L_1 \le 2L, \quad ML \asymp N, \quad u \le L \le v, \quad a_m,\ b_l \ll N^{\varepsilon}.$$

First we estimate the sums of second type. We have

$$W_2 \ll N^{\varepsilon} \sum_{M<m\le M_1} \left| \sum_{\substack{L<l\le L_1\\ N/2<ml\le N}} b_l \sum_{0<|k|\le H} c(k)e(\alpha mlk) \sum_{\substack{d\le D\\ d|ml+2\\ 2\nmid d}} \xi(d) \right|.$$

Applying the Cauchy inequality and (55), (61) we get

$$|W_2|^2 \ll N^{\varepsilon} M \sum_{M<m\le M_1} \left| \sum_{\substack{L<l\le L_1\\ N/2<ml\le N_1}} b_l \sum_{0<|k|\le H} c(k)e(\alpha mlk) \sum_{\substack{d\le D\\ d|ml+2\\ 2\nmid d}} \xi(d) \right|^2$$

$$\ll N^{\varepsilon} M \sum_{\substack{d_1,d_2\le D\\ 2\nmid d_1d_2}} \sum_{0<k_1,k_2\le H} \sum_{L<l_1,l_2\le L_1} |V|,$$

where

$$V = \sum_{\substack{M' < m \le M_1' \\ l_i m + 2 \equiv 0 (d_i),\, i=1,2}} e(\alpha m (k_1 l_1 - k_2 l_2)),$$

$$M' = \max\left\{ \frac{N}{2l_1},\, \frac{N}{2l_2},\, M \right\},\; M_1' = \min\left\{ \frac{N}{l_1},\, \frac{N}{l_2},\, M_1 \right\}.$$

If the system of congruences

$$(62) \qquad \left| \begin{array}{l} l_1 m + 2 \equiv 0(d_1) \\ l_2 m + 2 \equiv 0(d_2). \end{array} \right.$$

has no solution then $V = 0$. Assume that the system (62) has a solution. Then there exist an integer $f = f(l_1,\, l_2,\, d_1,\, d_2)$ such that (62) is equivalent to $m \equiv f([d_1,\, d_2])$ and therefore

$$\begin{aligned} V &= \sum_{\substack{M' < m \le M_1' \\ m \equiv f([d_1,\, d_2])}} e(\alpha m (k_1 l_1 - k_2 l_2)) \\ &= e(\alpha f (k_1 l_1 - k_2 l_2)) \sum_{\frac{M' - f}{[d_1,\, d_2]} < s \le \frac{M_1' - f}{[d_1,\, d_2]}} e(\alpha s [d_1,\, d_2](k_1 l_1 - k_2 l_2)). \end{aligned}$$

From (5), (54), (59), (61) it follows that

$$(63) \qquad M \gg \frac{N}{v} \gg D^2.$$

Applying Lemma 4 from [11], ch. 6, §2, we get

(64)

$$V \ll \begin{cases} \dfrac{M}{[d_1,\, d_2]}\,, & \text{if } k_1 l_1 = k_2 l_2, \\ \min\left\{ \dfrac{M}{[d_1,\, d_2]},\, \dfrac{1}{||\alpha(k_1 l_1 - k_2 l_2)[d_1,\, d_2]||} \right\}, & \text{if } k_1 l_1 \ne k_2 l_2. \end{cases}$$

Therefore

$$(65) \qquad |W_2|^2 \ll N^{\varepsilon} M \bigg(M V_1 + V_2 \bigg),$$

where

$$V_1 = \sum_{d_1,\, d_2 \le D} \frac{1}{[d_1,\, d_2]} \sum_{0 < k_1,\, k_2 \le H} \sum_{\substack{L < l_1,\, l_2 \le L_1 \\ k_1 l_1 = k_2 l_2}} 1,$$

$$V_2 = \sum_{d_1,\, d_2 \le D} \sum_{0 < k_1,\, k_2 \le H} \sum_{\substack{L < l_1,\, l_2 \le L_1 \\ k_1 l_1 \ne k_2 l_2}} \min\left\{ \frac{M}{[d_1,\, d_2]},\, \frac{1}{||\alpha(k_1 l_1 - k_2 l_2)[d_1,\, d_2]||} \right\}.$$

It is clear that

(66)
$$V_1 \ll \sum_{h\le D^2} \frac{1}{h} \sum_{[d_1,\, d_2]=h} 1 \sum_{n\le 2HL} \tau^2(n) \ll N^\varepsilon HL \sum_{h\le D^2} \frac{\tau^2(h)}{h} \ll N^\varepsilon HL.$$

Consider V_2 we have

$$V_2 \ll \sum_{h\le D^2} \tau^2(h) \sum_{0<|r|\le 2HL} \min\left\{\frac{M}{h},\, \frac{1}{||\alpha r h||}\right\} \sum_{\substack{0<n_1,\, n_2\le 2HL\\ n_1-n_2=r}} \tau(n_1)\tau(n_2)$$
$$\ll N^\varepsilon HL \sum_{h\le D^2} \sum_{0<r\le 2HL} \min\left\{\frac{M}{h},\, \frac{1}{||\alpha r h||}\right\}$$
$$\ll N^\varepsilon HL \sum_{m\le 2D^2 HL} \min\left\{\frac{HLM}{m},\, \frac{1}{||\alpha m||}\right\}.$$

Since $M \gg D^2$ (see (63)) we can apply Lemma 2.2 from [18], ch. 2, §2.1 and get

$$V_2 \ll N^\varepsilon\left(\frac{H^2L^2M}{Q} + D^2H^2L^2 + HLQ\right). \tag{67}$$

From (61), (65) – (67) we obtain

$$|W_2|^2 \ll N^\varepsilon\left(HLM^2 + \frac{H^2L^2M^2}{Q} + D^2H^2L^2M + HLMQ\right)$$
$$\ll N^\varepsilon\left(\frac{HN^2}{u} + \frac{H^2N^2}{Q} + D^2H^2Nv + HNQ\right).$$

Hence

$$W_2 \ll N^\varepsilon\left(\frac{H^{\frac12}N}{u^{\frac12}} + \frac{HN}{Q^{\frac12}} + DHN^{\frac12}v^{\frac12} + H^{\frac12}N^{\frac12}Q^{\frac12}\right). \tag{68}$$

Now we shall estimate the sums of the first type. Using (55), (60) we get

$$W_1 \ll N^\varepsilon \sum_{\substack{d\le D\\ 2\nmid d}} \sum_{0<k\le H} \sum_{M<m\le M_1} |U|, \tag{69}$$

where

$$U = \sum_{\substack{L'<l\le L_1'\\ ml+2\equiv 0(d)}} e(\alpha k m l), \tag{70}$$

$$L' = \max\left\{L,\, \frac{N}{2m}\right\},\quad L_1' = \min\left\{L_1,\, \frac{N}{m}\right\}.$$

If $(m,\, d) > 1$ then the sum U is empty. Suppose now that $(m,\, d) = 1$. Then the congruence $ml + 2 \equiv 0(d)$ is equivalent to $l \equiv l_0(d)$ for some integer $l_0 = l_0(m,d)$. Hence we may write U in the form

$$U = e(\alpha k m l_0) \sum_{\frac{L'-l_0}{d} < s \le \frac{L_1'+l_0}{d}} e(\alpha k m s d).$$

Using Lemma 4 from [11], ch. 6, §2 we get

$$U \ll \min\left\{\frac{N}{md}, \frac{1}{||\alpha k m d||}\right\},$$

consequently

$$\begin{aligned} W_1 &\ll N^{\varepsilon} \sum_{d\le D} \sum_{k\le H} \sum_{M<m\le M_1} \min\left\{\frac{N}{md}, \frac{1}{||\alpha k m d||}\right\} \\ &\ll N^{\varepsilon} \sum_{n\le 2MD} \sum_{k\le H} \min\left\{\frac{N}{n}, \frac{1}{||\alpha k n||}\right\} \\ &\ll N^{\varepsilon} \sum_{n\le 2MD} \sum_{k\le H} \min\left\{\frac{HN}{kn}, \frac{1}{||\alpha k n||}\right\} \\ &\ll N^{\varepsilon} \sum_{s\le 2MDH} \min\left\{\frac{NH}{s}, \frac{1}{||\alpha s||}\right\}. \end{aligned}$$

Using (54), (59), (60) we see that we may apply again Lemma 2.2, [18], ch. 2, §2.1 and we get

$$W_1 \ll N^{\varepsilon}\left(\frac{HN}{Q} + \frac{DHN}{w} + Q\right). \tag{71}$$

We consider the sum W_1' in the same manner and we find

$$W_1' \ll N^{\varepsilon}\left(\frac{HN}{Q} + \frac{DHN}{w} + Q\right). \tag{72}$$

From (68), (71) and (72) we obtain

$$W \ll N^{\varepsilon}\left(\frac{H^{\frac{1}{2}}N}{u^{\frac{1}{2}}} + \frac{HN}{Q^{\frac{1}{2}}} + DHN^{\frac{1}{2}}v^{\frac{1}{2}} + H^{\frac{1}{2}}N^{\frac{1}{2}}Q^{\frac{1}{2}} + \frac{DHN}{w} + Q\right).$$

We choose

$$N = Q^{\frac{2}{1+\theta}} \tag{73}$$

and having in mind (4), (54), (59) we obtain

$$W \ll N^{1+\frac{\theta}{2}-\frac{\delta}{4}} + N^{\frac{3(1+\theta)}{4}} + N^{\frac{2}{3}+\delta+\theta} \ll N^{1-\varpi}$$

for some small constant $\varpi > 0$. This proves our Lemma.

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